A Re-examination of the Joint Mortality Functions

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Abstract.

Mortality analysis involving multiple lives is easily one of the more complicated aspects in the theory of life contingencies. In this paper, we re-investigate joint mortality functions and in particular, we examine an assertion that relates the joint-life and last-survivor random variables. This common assertion states that the sum of the lifetimes of the joint-life and the last-survivor statuses is equal to the sum of the lifetimes of the single statuses. However, we show that this assertion is not precisely correct. We therefore offer a modification to the definitions of the statuses so that this common assertion holds.

1. Introduction.

The equation relating the joint-life and last-survivor future lifetime random variables is given as

\[ T(xy) + \overline{T(xy)} = T(x) + T(y), \]

without any independence assumption. This assertion relies on certain critical, yet unstated, assumptions. In Section 4, we will present a modified version of (1) with proper assumptions. Since many formulae rely on this assertion, a number of basic relationships in multiple life mortality functions turn out to be wrong. For instance,
are not valid without certain independence assumptions. (All necessary notations and definitions are given in the following section).

Since these relations are used to price joint life insurance products, it is important to investigate assumptions that make these relations hold, and when these assumptions do not hold, it is important to understand the correct relations. In Section 3 examples are given to illustrate that these equalities do not hold. In Section 4 the notations are examined in greater detail, and valid equalities are given.

2. Notations and definitions.

We follow closely the notations in the textbook, Bowers, et al. (1997). We put the context in terms of spousal mortality.

Let

$X$ be the random variable representing female’s age at death,

$Y$ be the random variable representing male’s age at death,

$(x)$ is the status denoting a person-aged-$x$,

$(xy)$ is the joint-life status that survives as long as both $(x)$ and $(y)$ survive,

$(\overline{xy})$ is the joint last-survivor status that exists as long as at least one of $(x)$ and $(y)$ is alive, and

$T(u)$ is the future lifetime for status $(u)$. 

(2) $p_{xy} + p_{\overline{xy}} = p_x + p_y$

and

(3) $A_{xy} + A_{\overline{xy}} = A_x + A_y$
We define, for a status \((u)\), the conditional survival function \(t_p^u\) and the net single premium \(A_u\) for life insurance policy that pays at the end of the year the status \((u)\) fails.

The annual interest rate is denoted by \(i\) and \(v = \frac{1}{1+i}\).

\[
_t p^x = P(T(x) > t) = P(X > x + t | X > x),
\]

\[
_t p^y = P(T(y) > t) = P(Y > y + t | Y > y),
\]

\[
_t p^{xy} = P(T(xy) > t) = P(X > x + t \text{ and } Y > y + t | X > x, Y > y),
\]

\[
_t p^{\overline{xy}} = P(T(\overline{xy}) > t) = P(X > x + t \text{ or } Y > y + t | X > x, Y > y),
\]

and

\[
A_u = \sum_k v^{k+1}(k p^u_k - k + 1 p^u_k), \text{ where } u \text{ represents a status } x, y, xy, \text{ or } \overline{xy}.
\]

3. Examples.

The following are counter-examples that show equalities (2) and (3) do not hold.

**Example 1.** \(\_t p^{xy} + \_t p^{\overline{xy}} \neq _t p^x + _t p^y\)

Consider a pair of animals from a purely fictional species. Shortly after birth they mate for life. A probability breakdown of their ages at death is given in the table below (\(K_m\) is the curtate future lifetime of a newborn male animal, etc.):

<table>
<thead>
<tr>
<th>(\text{P}(K_i = 0))</th>
<th>(\text{P}(K_i = 1))</th>
<th>(\text{P}(K_i = 2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{P}(K_m = 0))</td>
<td>.10</td>
<td>.10</td>
</tr>
<tr>
<td>(\text{P}(K_m = 1))</td>
<td>.10</td>
<td>.10</td>
</tr>
<tr>
<td>(\text{P}(K_m = 2))</td>
<td>.10</td>
<td>.10</td>
</tr>
<tr>
<td></td>
<td>.30</td>
<td>.30</td>
</tr>
</tbody>
</table>
Thus, for a newly mated pair, the probability that the male dies in its third year \((K_m = 2)\) while the female dies in its third year \((K_f = 2)\) is .20. Consider a pair, which have both survived their first year. Then we have the following:

\[
P_{1:1} = P(\text{both survive at least one more year (given they survived thus far))}
= \frac{.20}{.50} = \frac{2}{5}
\]

\[
P_{1:1}^m = P(\text{at least one survives one or more years (given they survived thus far))}
= \frac{.40}{.50} = \frac{4}{5}
\]

\[
P_{1}^m = P(\text{the male survives at least one more year (given it survived thus far))}
= \frac{.40}{.70} = \frac{4}{7}
\]

\[
P_{1}^f = P(\text{the female survive at least one more year (given it survived thus far))}
= \frac{.40}{.70} = \frac{4}{7}
\]

Clearly, \(\frac{2}{5} + \frac{4}{5} \neq \frac{4}{7} + \frac{4}{7}\), so \(P_{xy} + P_{xy}^m \neq P_x^m + P_y\).

**Example 2.** \((A_{xy} \neq A_x + A_y - A_{xy})\)

For calculating last-survivor actuarial present values, \(A_{xy} = A_x + A_y - A_{xy}\) is commonly used. However, it turns out that this formula does not hold in general.

Consider a copula model (e.g., Hougaard’s copula with Weibull marginals):

Let the bivariate survival function \(S(x, y) = P(X > x \text{ and } Y > y) = C(S_1(x), S_2(y))\), where

\[
S_j(x) = P(X > x) = \exp\left(-\left(\frac{x}{m_j}\right)^{\frac{m_j}{\sigma_j}}\right) \text{ for } j = 1, 2, \text{ and }
\]
\[ C(u,v) = \exp \left( - \left[ (-\ln u)^{\alpha} + (-\ln v)^{\alpha} \right]^{1/\alpha} \right). \]

Note that \( C(u,1) = u \) and \( C(1,v) = v \), thus the marginal survival functions derived from the bivariate survival function \( S(x,y) \) coincide with \( S_1(x) \) and \( S_2(y) \), respectively.

Let \( X \) be the "female age at death" random variable, and let \( Y \) be the "male age at death" random variable as defined earlier. We consider the model with \( m_1 = 89.51, \sigma_1 = 8.99, m_2 = 85.98, \sigma_2 = 11.24, \alpha = 1.638 \) (these parameters come from using the maximum likelihood method on experience data for joint annuity contracts from an insurance company – see Youn and Shemyakin, 1999) and compute the net single premiums \( A^f_{65}, A^m_{70}, A_{65:70}, \) and \( A^{65:70}_{65:70} \) (\( A^m_{65:70} \) and \( A^{65:70}_{65:70} \) were computed using the bivariate survival function with all the computations done on Mathematica).

With \( i = .05 \), we have the following results:

\[ A^f_{65} = 0.380867, A^m_{70} = 0.504678, A_{65:70} = 0.525265, \text{ and} \]
\[ A^{65:70}_{65:70} = 0.327022. \]

The expression \( A^f_{65} + A^m_{70} - A_{65:70} = 0.360280 \) yields an error of more than 10% !

The following table demonstrates the values of the ratio \( \frac{A^f_{65} + A^m_{70} - A_{65:70}}{A^{65:70}_{65:70}} \) for various values of the association parameter \( \alpha \) and the interest rate \( i \).

**Table 1.** The ratio \( \frac{A^f_{65} + A^m_{70} - A_{65:70}}{A^{65:70}_{65:70}} \) for various values of \( \alpha \) and \( i \).
<table>
<thead>
<tr>
<th>Interest rate</th>
<th>3%</th>
<th>6%</th>
<th>9%</th>
<th>12%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Association $\alpha$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1.5</td>
<td>1.04856</td>
<td>1.11305</td>
<td>1.19543</td>
<td>1.29657</td>
</tr>
<tr>
<td>2</td>
<td>1.06650</td>
<td>1.15605</td>
<td>1.27165</td>
<td>1.41499</td>
</tr>
<tr>
<td>3</td>
<td>1.08126</td>
<td>1.19357</td>
<td>1.34269</td>
<td>1.53365</td>
</tr>
</tbody>
</table>

Association $\alpha = 1$ corresponds to independence between male and female lives. Higher values of association and higher interest rates bring about a substantial discrepancy between the exact value $A_{65:70}$ and its approximation $A_{65}^f + A_{70}^m - A_{65:70}$.

### 4. Propositions and Analysis.

We present a modified version of equality (1) $T_{(xy)} + T_{(xy)} = T(x) + T(y)$ that correctly relates the lifetimes of joint-life and last-survivor statuses to the lifetimes of single statuses.

First, let us closely examine why equality (2) $p_{xy} + p_{xy} = p_x + p_y$ is not true in general.

We start with abstract probability arguments, rather than mortality arguments. The reason is that many of us are so familiar with the equality (2), it is worthwhile to step back a little.

Let $A$, $B$, $C$, and $D$ be random events. Then the conditional probabilities of $A \cap B$ and $A \cap B$ given $C \cap D$ are related by the equation
(a) \[\begin{align*}
P(A \cup B \mid C \cap D) + P(A \cap B \mid C \cap D) &= P(A \mid C \cap D) + P(B \mid C \cap D).
\end{align*}\]

Note that the same identical condition (that \(C \cap D\) be true) is present in each of the four probabilities; to change some of these conditions would create a statement which is not always true. For example,

(b) \[\begin{align*}
P(A \cup B \mid C \cap D) + P(A \cap B \mid C \cap D) &\neq P(A \mid C) + P(B \mid D)
\end{align*}\]

in the general case (although it may be fairly close in many cases).

We will now translate this result into actuarial terms.

Let \(A = \{X > x + t\}\), \(B = \{Y > y + t\}\), \(C = \{X > x\}\), and \(D = \{Y > y\}\).

Then \(P(A \cup B \mid C \cap D) = t p^{-xy} \), \(P(A \cap B \mid C \cap D) = t p_{xy} \), \(P(A \mid C) = t p_x \), and

\[P(B \mid D) = t p_y,\]

so inequality (b) shows us that \(t p_{xy} + t p^{-xy} \neq t p_x + t p_y\).

We see that the equality (2) does not hold because \(t p_x\) makes no assumptions on the current status of \(Y\) and \(t p_y\) likewise makes no assumptions on whether \(X\) survives to age \(x\) or not, while \(t p_{xy}\) and \(t p^{-xy}\) require both \(X > x\) and \(Y > y\) as conditions. The same is true for the relation between \(T(x), T(y), T(xy)\) and \(T(xy)\). To define \(T(x)\), one makes no assumptions on the current status of \(Y\) and, to define \(T(y)\), one makes no assumptions on whether \(X\) survives to age \(x\) or not, but to define \(T(xy)\) and \(T(xy)\), both \(X > x\) and \(Y > y\) are required. In order to relate joint life functions with single life functions, one needs to consider single statuses in a joint context with spousal information.
Notation. We introduce \((x||y)\) to denote a status of a person-aged-\(x\) whose spouse is a person-aged-\(y\). Thus the future lifetime random variables for the statuses \((x||y)\) and \((y||x)\) are given by

\[T(x||y) = X-x, \text{ defined when both } X>x \text{ and } Y>y, \text{ and}\]

\[T(y||x) = Y-y, \text{ defined when both } X>x \text{ and } Y>y.\]

Now we can state that

\[T(xy) = \min(X-x, Y-y), \text{ defined when both } X>x \text{ and } Y>y, \text{ and equals } \min(T(x||y), T(y||x)),\]

and

\[T(xy) = \max(X-x, Y-y), \text{ defined when both } X>x \text{ and } Y>y, \text{ and equals } \max(T(x||y), T(y||x)).\]

The conditional survival functions for the statuses \((x||y)\) and \((y||x)\) would become

\[tP_{x||y} = P(T(x \mid y) > t) = P(X > x + t \mid X > x, Y > y) \text{ and}\]

\[tP_{y||x} = P(T(y \mid x) > k) = P(Y > y + t \mid X > x, Y > y).\]

We now can properly state

\[tP_{xy} + tP_{xy} = tP_{x||y} + tP_{y||x}.\]

We state the following proposition without any further proof.

Proposition 1.

\[T(xy) = \min(T(x||y), T(y||x)),\]

\[T(xy) = \max(T(x||y), T(y||x)), \text{ and}\]

\[T(xy) + T(xy) = T(x||y) + T(y||x).\]
We want to note that, while \( T(x||y), T(x||y), T(xy), \) and \( T_{xy} \) are all defined on the common domain \( X>0, Y>0, T(x) \) and \( T(y) \) are not: in a joint context, \( T(x) \) is defined when \( X>0, Y>0 \) and \( T(y) \) is defined when \( X>0, Y>0 \). In such a context, \( T(x) \) and \( T(y) \) do not have a common domain and cannot be added as random variables. Thus, (7) may be considered as a correction of (1).

**Proposition 2.** There exist random variables \( X \) and \( Y \), that for some \( x, y, \) and \( t \)

\[
\begin{align*}
(8) & \quad t \, p_{xy} + t \, p_{x} \neq t \, p_{x} + t \, p_{y} \\
(9) & \quad A_{xy} + A_{xy} \neq A_{x} + A_{y}.
\end{align*}
\]

**Proof.** Statement (8) is demonstrated by Example 1. Statement (9) is demonstrated by Example 2.

We note that expressing \( _t \, p_x \) and \( _t \, p_y \) in a joint context would be

\[
\begin{align*}
(10) & \quad _t \, p_x = P(T(x)>t) = P(X>x+t \mid X>x, Y>0) \\
(11) & \quad _t \, p_y = P(T(y)>t) = P(Y>y+t \mid Y>y, X>0).
\end{align*}
\]

In the same vein, identifying \( T(x) \) with \( T(x||0) \) would be mathematically correct, although its interpretation may seem unnatural.

In the Appendix, we examine the difference between \( T(x) \) and \( T(x||y) \) as they relate to life insurance premiums. A further treatment of insurance premiums with spousal status is

If lives $X$ and $Y$ are indeed independent, the statements (2) and (3) become true. Unfortunately, recent studies show that survivals of pairs of husbands and wives are not independent. (See Annuity Valuation with Dependent Mortality by Frees, et. al.)

The Proposition 2 is not as abstract or trivial as it might seem. An assumption of equality in (2) and (3) is the basis of many important relationships in multiple life mortality function theory. The Third Examination of the Society of Actuaries tests knowledge of multiple life mortality functions (among other topics) and historically has made use of these formulae.

5. Conclusion.

What kind of independence assumption is required to create equality in (8)? The answer is, as the conditions in the expressions (10) and (11) indicate, the mortality rate of the female or the male should not depend on whether they have a surviving spouse or not, nor on the surviving spouse’s age. This is generally assumed in practice. Insurance companies do not classify insurers according to whether one has a surviving spouse or not, nor to spouse’s age. It is worth noting that, according to a copula model as illustrated by Tables 2-4 in the Appendix, the life insurance premiums with spousal status classification are lower than those without the classification. The percentage differences are higher for older spouses and for higher interest rates.
References


Appendix

To illustrate an effect the difference between $T(x)$ and $T(x \parallel y)$ has in pricing life insurance, we compare insurance premiums based on $T(x)$ and $T(x \parallel y)$.

We note that $k p_x = P(T(x) > k) = P(X > x + k \mid X > x)$, $A_x = \sum v^{k+1}(k p_x - k+1 p_x)$ and $k p_{x\parallel y} = P(T(x \parallel y) > k) = P(X > x + k \mid X > x, Y > y)$, and compute the insurance premium with spousal status $A_{x\parallel y} = \sum v^{k+1}(k p_{x\parallel y} - k+1 p_{x\parallel y})$.

The following tables demonstrate the ratio $A_{x\parallel y} / A_x$ between the premium values for females age 50-80 with and without spousal classification, spousal ages allowed to vary. Thus, $x$ denotes female’s age and $y$ denotes her spouse’s age. Computations were preformed according to Hougaard’s copula model with Weibull marginals $m_1 = 89.51$, $\sigma_1 = 8.99$, $m_2 = 85.98$, $\sigma_2 = 11.24$, and $\alpha = 1.638$. Interest rate varies from 3% to 7%.

**Table 2.** The ratio $A_{x\parallel y} / A_x$ with interest rate 3%.

<table>
<thead>
<tr>
<th>Female Age</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>.98811</td>
<td>.96314</td>
<td>.92263</td>
<td>.87053</td>
</tr>
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<td>.88642</td>
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<td>70</td>
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<tr>
<td>80</td>
<td>.99985</td>
<td>.99852</td>
<td>.99093</td>
<td>.96669</td>
</tr>
</tbody>
</table>

**Table 3.** The ratio $A_{x\parallel y} / A_x$ with interest rate 5%.

<table>
<thead>
<tr>
<th>Female Age</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
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<tr>
<td>80</td>
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<td>.99756</td>
<td>.98511</td>
<td>.94586</td>
</tr>
</tbody>
</table>
Table 4. The ratio $A_{x||y}/A_x$ with interest rate 7%.

<table>
<thead>
<tr>
<th>Female Age</th>
<th>Male Age</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
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<tr>
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