Homothetic Functions

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is homothetic if it can be expressed as an increasing function of a function that is homogeneous of some positive degree; that is, for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$: there exists a homogeneous function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ and an increasing function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x_1, x_2, \ldots, x_n) = F(h(x_1, x_2, \ldots, x_n)).$$

Claim. A function is homothetic if and only if it can be expressed as a monotonic transformation of a linearly homogeneous function.

proof: Suppose $f(\cdot)$ is homothetic and assume the inner function $h(\cdot)$ is homogeneous of degree $r$ in $(x_1, x_2, \ldots, x_n)$. Define the outer function $G : \mathbb{R} \rightarrow \mathbb{R}$ as $G(y) = F(y^r)$ and the inner (homogeneous) function as $g(x_1, x_2, \ldots, x_n) = (h(x_1, x_2, \ldots, x_n))^{\frac{1}{r}}$. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and let $t > 0$ be arbitrary. Since

$$g(tx_1, tx_2, \ldots, tx_n) = (h(tx_1, tx_2, \ldots, tx_n))^{\frac{1}{r}}$$

$$= (t^r h(x_1, x_2, \ldots, x_n))^{\frac{1}{r}}$$

$$= t (h(x_1, x_2, \ldots, x_n))^{\frac{1}{r}}$$

$$= tg(x_1, x_2, \ldots, x_n),$$

$g(\cdot)$ is homogeneous of degree 1. Since

$$G'(y) = \frac{d}{dy} F(y^r) = F'(y^r)ry^{r-1} > 0,$$

$G(\cdot)$ is increasing (i.e., for $r > 0$). To complete the proof, note that

$$G(g(x_1, x_2, \ldots, x_n)) = F \left[ \left( \left( h(x_1, x_2, \ldots, x_n) \right)^r \right)^{\frac{1}{r}} \right] = F(h(x_1, x_2, \ldots, x_n)) = f(x_1, x_2, \ldots, x_n).$$