The Automorphic Heat Kernel: Spectral and Geometric Points of View

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Applications to Number Theory

- asymptotic formulas for spectra of $\Gamma \backslash G$
- relationship between $\eta$-invariants and closed geodesics
  - Moscovici-Stanton 1989
- zeta functions from heat Eisenstein series
- sup-norm bounds for automorphic forms
  - Jo-Kra04, Jo-Kra11, Ary16, Fr-Jo-Kra16, Ary-Bal18
- limit formulas, Weyl-type asymptotic for period integrals
  - Tsuzuki 2008, 2009
- ave. holo. QUE for afc cfms for quaternion algebras
  - Aryasomayajula-Balasubramanyam 2018
Wind-up heat kernel on G/K

- Gangolli 1968:
  - integral representation for heat kernel on G/K
  - explicit formula when G/K of complex type
  - wind-up by averaging over cocompact \( \Gamma \)
- Special cases: \( G/K = \mathbb{H}^d \), \( G = \text{SL}_n(\mathbb{C}) \), etc.
  - e.g. Fay 1977, Jorgenson-Lang 2009
- Convergence in general? (Existence?)
- Automorphic spectral expansion?
  - “conjectural” (Jorgenson-Lang 2009)
Our Approach

Spectral:

- using global automorphic Sobolev theory
- construct automorphic heat kernel via automorphic spectral expansion in terms of cusp forms, Eisenstein series, and residues of Eisenstein series
  - existence of automorphic heat kernel
- prove uniqueness (semigroup theory)
- prove $C^\infty$-convergence of automorphic spectral expansion and smoothness of automorphic heat kernel (for $t > 0$)

Geometric:

- use known bound on heat kernel on $G/K$
- wind-up: proof involves norm on $G$
1D Euclidean Heat Kernel

Heat kernel \( u : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R} \) satisfies:

\[
(\partial_t - \Delta) u = 0, \quad \lim_{t \to 0^+} u(x, t) = \delta.
\]

Apply Fourier transform \( \mathcal{F} \):

\[
(\partial_t + 4\pi^2 \xi^2) \mathcal{F}u = 0, \quad \lim_{t \to 0^+} (\mathcal{F}u)(\xi, t) = \mathcal{F}\delta = 1.
\]

Considering \( \xi \) as fixed, \( \mathcal{F}u(\xi, t) \) satisfies familiar IVP:

\[
\frac{dy}{dt} = -4\pi^2 \xi^2 y, \quad y(0) = 1 \quad \Rightarrow \quad y(t) = e^{-4\pi^2 \xi^2 t}
\]

Fourier inversion: \( u(x, t) = (4\pi t)^{-1/2} e^{-x^2/4t} \).
Automorphic Analogue

- $X = \Gamma \backslash G / K$, with $G$, red. or ss. Lie group, max. compact $K \subset G$, arithmetic $\Gamma \subset G$
- $\Delta$, Laplacian on $\Gamma \backslash G$ (the image of Casimir)
- $\delta$, automorphic delta distribution at $x_0 = \Gamma \cdot 1 \cdot K$

Want $u(x, t)$ on $X \times (0, \infty)$ satisfying

$$(\partial_t - \Delta) u = 0 \quad \text{and} \quad \lim_{t \to 0^+} u(x, t) = \delta$$

Apply spectral transform $\mathcal{F}$ to get IVP on spectral side

$$(\partial_t - \lambda \xi) \mathcal{F} u = 0 \quad \text{and} \quad \lim_{t \to 0^+} \mathcal{F} u(\xi, t) = \mathcal{F} \delta$$

Solve IVP: $\mathcal{F}(u, \xi) = \mathcal{F} \delta \cdot e^{\lambda \xi t}$; spectral inversion $\rightarrow u(x, t)$.
(Global Afc) Sobolev Theory

Physical Side \( \mathcal{X}, \Delta \)
differentiation

\[ \mathcal{F} \quad \mathcal{F}^{-1} \]

Spectral Side \( \Xi, \lambda \xi \)
multiplication

Solve differential equations by division!

- Only for Schwartz functions? \( \ldots \) \( L^2 \)-functions?
- Functions in (global afc) Sobolev spaces: \( H^s(X) \)

Other applications:
- lattice point counting in \( G/K \) (D. 2012)
- behavior of 4-loop supergraviton (Klinger-Logan, 2018)
Time Parameter

- View heat kernel as $H^s$-valued function of $t$.

$$U : (0, \infty) \to H^s(X)$$

- Limit as $t \to 0^+$ is in $H^s$-topology

- Strong differentiation (vs weak) w.r.t. $t$

- Translation Lemma
  - from physical side to spectral side and back
  - limits, weak and strong differentiability, differential equations for $U$ and $F \circ U$
Spectral Theory for $\text{SL}_2(\mathbb{Z}) \backslash \mathcal{H}$

Consider $X = \text{SL}_2(\mathbb{Z}) \backslash \mathcal{H}$, with Laplacian $\Delta = y^2 \left( \frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right)$.

Spectral inversion: eigenfunction expansion

$$f \overset{L^2}{=} \sum_F \langle f, F \rangle \cdot F + \langle f, \Phi_0 \rangle \cdot \Phi_0 + \frac{1}{4\pi i} \int_{\frac{1}{2} + i\mathbb{R}} \langle f, E_s \rangle \cdot E_s \ ds$$

where

- $F$ in o.n.b. of cusp forms,
- $\Phi_0$ is the constant automorphic form with unit $L^2$-norm,
- and $E_s$ is the real analytic Eisenstein series

Note: integrals are extensions by isometric isomorphisms of continuous linear functionals on $C_c^\infty(X)$. 
Automorphic Spectral Theory

Abbreviate (and generalize): denote elements of the spectral “basis” (cusp forms, Eisenstein series, residues of Eisenstein series) uniformly as \( \{ \Phi_\xi \}_{\xi \in \Xi} \).

\[
f = \int_\Xi \langle f, \Phi_\xi \rangle \cdot \Phi_\xi \, d\xi,
\]

View \( \Xi \) as a finite disjoint union of spaces of the form \( \mathbb{Z}^n \times \mathbb{R}^m \) with usual measures.
Automorphic Sobolev Spaces

Inner product $\langle \, , \, \rangle_s$ (for $0 \leq s \in \mathbb{Z}$) on $C_c^\infty(X)$ by

$$\langle \varphi, \psi \rangle_s = \langle (1 - \Delta)^s \varphi, \psi \rangle_{L^2}$$

Sobolev spaces:
- $H^s$ is Hilbert space completion of $C_c^\infty(X)$ w.r.t. topology induced by $\langle \, , \, \rangle_s$
- $H^{-s}$ is Hilbert space dual of $H^s$.

Note:
- $H^0 = L^2(X)$
- Nesting: $H^s \hookrightarrow H^{s-1}$ for all $s$. 
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Diff’n and Spectral Transform

\[
\begin{align*}
&\ldots \quad \mathcal{H}^+ s \quad \overset{(1-\Delta)}{\approx} \quad \mathcal{H}^+ s - 2 \quad \overset{(1-\Delta)}{\approx} \quad \ldots \\
&\mathcal{F} \quad \overset{\approx}{\rightarrow} \\
&\ldots \quad \mathcal{V}^+ s \quad \overset{\approx}{\times} (1-\lambda_{\xi}) \quad \overset{\approx}{\rightarrow} \\
&\mathcal{F} \quad \overset{\approx}{\rightarrow} \\
&\ldots \quad \mathcal{V}^+ s - 2 \quad \overset{\approx}{\times} (1-\lambda_{\xi}) \quad \overset{\approx}{\rightarrow} \\
\end{align*}
\]

- spectral transform \( \mathcal{F} : f \mapsto \langle f, \Phi_{\xi} \rangle \)
- \( \lambda_{\xi} \) is the \( \Delta \)-eigenvalue of \( \Phi_{\xi} \)
- weighted \( L^2 \)-space \( V^s \): \( f \in V^s \) means \( (1-\lambda_{\xi})^{s/2} f \in L^2(\Xi) \)

Note:
- \( \Delta \) nonpositive symmetric operator \( \Rightarrow \) \( \lambda_{\xi} \leq 0 \)
- \( \Lambda : \xi \mapsto \lambda_{\xi} \) is differentiable and of moderate growth by a pre-trace formula
Key Results

- Every $u \in H^s$ has a spectral expansion, converging in the $H^s$-topology.
- Global automorphic Sobolev embedding theorem
  - For $s > k + (\dim X)/2$, $H^s \hookrightarrow C^k$.
  - Implies: $H^\infty = C^\infty$
- Pretrace formula $\Rightarrow \delta \in H^s$ for every $s < -(\dim X/2)$
  \[ \delta = \int_{\Xi} \overline{\Phi_\xi(x_0)} \Phi_\xi \, d\xi \quad (\text{conv. in } H^s, s < -(\dim X/2)) \]
- Expected spectral coefficient for automorphic heat kernel:
  \[ \mathcal{F}\delta \cdot e^{\lambda_\xi t} = \overline{\Phi_\xi(x_0)} \cdot e^{\lambda_\xi t}. \]
Automorphic Heat Kernel

Let $\ell$ be the smallest integer strictly greater than $\dim X/2$.

We define an automorphic heat kernel to be a map $U : (0, \infty) \to H^{-\ell}(X)$ such that

1. $U$ satisfies the “initial condition,”

$$\lim_{t \to 0^+} U(t) = \delta \quad \text{in} \quad H^{-\ell}(X).$$

2. For some $s \leq -\ell - 2$, $U$ is strongly differentiable as an $H^s$-valued function and satisfies the “heat equation”, i.e. for $t > 0$,

$$U'(t) - \Delta U(t) = 0 \quad \text{in} \quad H^s(X).$$
Existence; Spectral Expansion

For $t \geq 0$, let $U(t) = \int_{\Xi} \overline{\Phi}_\xi(x_0) \cdot e^{\lambda_\xi t} \cdot \Phi_\xi \, d\xi$.

Theorem (1)

1. For $t \geq 0$, $U(t) \in H^{-\ell}$.

2. $\lim_{t \to 0^+} U(t) = \delta$ in the topology of $H^{-\ell}$.

3. For $s \leq -\ell - 5$, viewing $U$ as a $H^s$-valued function, $U$ is strongly $C^1$ on $(0, \infty)$ and satisfies the “heat equation,” i.e. for $t > 0$,

$$\frac{d}{dt} U(t) - \Delta U(t) = 0 \quad \text{in} \ H^s,$$

where $\frac{d}{dt} U$ denotes the strong derivative of $U$.

In particular, $U(t)$ is an automorphic heat kernel.
The Automorphic Heat Kernel

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Introduction

Spectral Solution

Global Afc Sobolev Theory

Spectral Construction

Uniqueness

Smoothness

Example

Geometric Solution

Proof outline

For $t \geq 0$, let $\tilde{U}(t): \xi \mapsto \Phi_\xi(x_0) e^{\lambda \xi t}$.

Prove that:

- $\tilde{U}$ takes values in $V^{-\ell}$.
- $\tilde{U}(t) \to \mathcal{F}\delta$ in $V^{-\ell}$ as $t \to 0^+$.
- $\tilde{U}$ is weakly $C^k$ when viewed as a $V^{-\ell-2N}$-valued function, for $N > k$
  - weakly $C^2$ when viewed as $V^{-\ell-5}$-valued function
  - strongly $C^1$ (by weak-to-strong diff. principle)
- $\tilde{U}$ satisfies the (strong) differential equation

$$\frac{d}{dt} Y(t) = \lambda \xi Y(t)$$

when viewed as a $V^{-\ell-5}$-valued function.

Use the translation lemma.
Uniqueness and improved differentiability

Theorem (2)

1. The automorphic heat kernel constructed in Theorem 1 is the unique automorphic heat kernel.

2. It is strongly $C^1$ as a $H^{-\ell-2}$-valued function on $[0, \infty)$.

NB: By Thm 1, $U$ is strongly $C^1$ as a $H^{-\ell-5}$-valued function.

Idea of proof: use semigroup theory to prove uniqueness of solution to IVP on spectral side.
Proof outline

Prove uniqueness of $\tilde{U}$ as (suitable) $V^s$-valued solution to IVP:

$$\frac{d}{dt} Y(t) = \lambda_\xi Y(t), \quad Y(0) = \Phi_\xi(x_0)$$

Multiplication by $\lambda_\xi$: $V^{s+2} \to V^s$

- continuous linear map when $V^{s+2}$ and $V^s$ have their own (different) topologies as (differently) weighted $L^2$ spaces
- change perspective: view $V^{s+2}$ as subspace of $V^s$
  - wreak continuity
  - unbounded operator on Hilbert space $V^s$
  - prove: densely defined, negative, self-adjoint
  - prove: resolvent set contains $(0, \infty)$
  - infinitesimal generator of a SCCSG (Hille-Yosida)
Abstract Cauchy Problem

Have shown that multiplication by $\lambda \xi$ is the infinitesimal generator of a SCCSG, so can use the following:

**Proposition**

Let $G(t)$ be a SCCG in a Banach space $V$, let $A$ be the infinitesimal generator for $G(t)$ with domain $D$, and $v_0 \in D$. Then there is a unique function $[0, \infty) \to V$ that (i) is strongly continuous on $[0, \infty)$, (ii) is strongly differentiable on $(0, \infty)$, (iii) takes values in $D$, and (iv) solves the initial value problem,

$$\frac{d}{dt} Y(t) = A Y(t) ; \quad Y(0) = v_0 .$$

Moreover, the solution is strongly $C^1$ on $[0, \infty)$. 
Smoothness

Theorem (3)

For $t > 0$, the automorphic heat kernel lies in $C^\infty(X)$, and its automorphic spectral expansion

$$U(t) = \int_\Xi \Phi_\xi(x_0) \cdot e^{\lambda_\xi t} \cdot \Phi_\xi \, d\xi$$

converges in the $C^\infty(X)$-topology.

Proof outline.

- prove: $\tilde{U}(t) \in V^s$ for all $s$
- thus $U(t) \in H^s(X)$ for all $s$
- global automorphic Sobolev embedding theorem
  $\Rightarrow U(t) \in C^k$ for all $k$
Example: $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$

**Corollary**

The unique automorphic heat kernel on $X = \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ is:

$$U(t) = \sum_{F} \overline{F}(x_0) e^{\lambda_{F} t} \cdot F + \overline{\Phi}_0(x_0) \cdot \Phi_0$$

$$+ \frac{1}{4\pi i} \int_{\frac{1}{2} + i\mathbb{R}} \overline{E}_s(x_0) e^{s(s-1)t} \cdot E_s \, ds$$

For $t > 0$, $U(t)$ is a smooth function on $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$, and its spectral expansion converges to it in the $C^\infty$-topology.
Geometric Perspective

Bi-K-invariant heat kernel on $G$, conn. ss. Lie, finite center.

- Construct via spherical inversion:

$$h_t(a) = \int_{W \setminus a^*} e^{-t(\|\lambda\|^2 + |\rho|^2)} \varphi_\lambda(a) |c(\lambda)|^{-2} d\lambda.$$ 

- For $G$ complex, $h_t(a)$ is a constant times:

$$(4\pi t)^{-n/2} e^{-t|\rho|^2} \prod_{\alpha \in \Sigma^+} \frac{\alpha(\log a)}{2 \sinh \alpha(\log a)} e^{-|\log a|^2/4t}.$$ 

For $\Gamma$ discrete, try winding up: $\sum_{\gamma \in \Gamma} h_t(\gamma g)$. 

Convergence? What kind of function on $\Gamma \setminus G/K$?
Geometric Construction

Theorem (4)

For $t > 0$, the Poincaré series $\sum_{\gamma \in \Gamma} h_t(\gamma g)$

- converges absolutely and uniformly on compacts,
- is of moderate growth, and
- is square integrable mod $\Gamma$.

Proof uses:

- norms on groups arguments for convergence etc. (Garrett)
- non-trival (but not sharp) bound for $h_t$ (Anker et al.)
Norms on groups

$G$, countably based, locally compact, Hausdorff, unimodular
topological group $G$ with compact subgroup $K$

Norm on $G$, a continuous function $\| \cdot \| : G \to (0, \infty)$ with:

- $\| \text{id}_G \| = 1$, where $\text{id}_G$ is the identity element in $G$,
- $\| g \| \geq 1$, for all $g$ in $G$,
- $\| g \| = \| g^{-1} \|$, for all $g$ in $G$,
- submultiplicativity: $\| gh \| \leq \| g \| \cdot \| h \|$, for all $g, h$ in $G$,
- $K$-invariance: $\| kgk' \| = \| g \|$, all $g$ in $G$, $k$, $k'$ in $K$,
- integrability: for some $r_0 \geq 0$,
\[
\int_G \| g \|^{-r} \, dg < \infty \quad (r > r_0).
\]
Poincaré series

- $G$ as in previous, $\Gamma$ discrete subgroup
- Norm $\| \cdot \|$ on $G$ with integrability exponent $r_0$.
- For suitable $f : G \to \mathbb{C}$, have Poincaré series:

$$P\tilde{e}_f(g) = \sum_{\gamma \in \Gamma} f(\gamma g)$$

Theorem (Garrett; see 2010 paper with Diaconu)

- If $|f(g)| \ll \|g\|^{-r}$ for some $r > r_0$, then the associated Poincaré series converges absolutely and uniformly on compact sets to a function of moderate growth.
- If $|f(g)| \ll \|g\|^{-2r}$ for some $r > r_0$, then $P\tilde{e}_f$ is square integrable modulo $\Gamma$. 
Outline of proof of Theorem 4

Prove:

- $\|g\| = \|kak'\| = e^{\log a}$ is a norm on $G$,
- with integrability expt: $r_0 = \sum_{\alpha \in \Sigma^+} m_\alpha |\alpha|$.

To apply Garrett’s theorem, want:

$$h_t(a) \ll e^{-2r|\log a|}, \text{ some } r > r_0.$$ 

Non-trivial (but not sharp) bound (Anker et. al) suffices:

$$h_t(a) \ll t^{-n/2} e^{-|\rho|^2 t - \langle \rho, \log a \rangle - |\log a|^2/4t} \quad (t > 0).$$
Poincaré series gives a **weak** automorphic heat kernel.

- Poincaré series is in \( L^2(X) = H^0(X) \subset H^{-\ell}(X) \).
- Limit as \( t \to 0^+ \) approaches \( \delta \) weakly in \( H^{-\ell} \).
- Weakly differentiable as \( H^s \)-valued function of \( t \) and satisfies weak version of automorphic heat equation.

Is the Poincaré series an automorphic heat kernel, as we have defined it? **If so**, can apply uniqueness theorem:

- it has the automorphic spectral expansion stated earlier,
- converging in the \( C^\infty \) topology for \( (t > 0) \),
- so for \( t > 0 \) is a smooth function on \( X \).
Thank you for your attention!