Cost Minimization with a CES Production Function

Consider a firm producing a single output with two productive inputs. Assume the firm’s production function has the Constant Elasticity of Substitution (CES) form.

\[ y = f(x_1, x_2) = (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}} \]

where

- \( y \) is the amount of output produced
- \( x_1 \) is the amount of labor employed
- \( x_2 \) is the amount of capital employed

\[ \alpha_1 > 0, \quad \alpha_2 > 0, \quad \alpha_1 + \alpha_2 = 1 \quad -\infty < \rho < 1. \]

**Primal Problem:**

Minimize \( w_1 x_1 + w_2 x_2 \)

Subject to: \( y_0 = (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}} \)

Form the Lagrangian

\[ \mathcal{L}(x_1, x_2; w_1, w_2, y_0) = w_1 x_1 + w_2 x_2 + \lambda \left( y_0 - (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}} \right). \]

The first-order conditions are

\[ \begin{align*}
\mathcal{L}_1 &= \frac{\partial \mathcal{L}}{\partial x_1} = w_1 - \lambda \left( \frac{1}{\rho} \right) (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho} - 1} \left( \rho \alpha_1 x_1^{\rho - 1} \right) = 0 \\
\mathcal{L}_2 &= \frac{\partial \mathcal{L}}{\partial x_2} = w_2 - \lambda \left( \frac{1}{\rho} \right) (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho} - 1} \left( \rho \alpha_2 x_2^{\rho - 1} \right) = 0 \\
\mathcal{L}_\lambda &= \frac{\partial \mathcal{L}}{\partial \lambda} = y_0 - (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}} = 0.
\end{align*} \]

Solve the first equation for \( w_1 \), the second equation for \( w_2 \) and divide \( w_1 \) by \( w_2 \) to obtain the condition that the iso-quant is tangent to the iso-cost.

\[ \frac{w_1}{w_2} = \frac{\alpha_1 x_1^{\rho - 1}}{\alpha_2 x_2^{\rho - 1}}. \]

Solve this equation for \( x_2 \), maintaining the symmetry of the problem.

\[ x_2 = \left( \frac{w_1}{\alpha_1} \right)^{\frac{1}{\rho}} \left( \frac{w_2}{\alpha_2} \right)^{\frac{1}{\rho}} x_1 \]

Raise both sides of this equation to the power \( \rho \) and substitute \( x_2^\rho \) into the constraint.

\[ y_0 = \left( \alpha_1 x_1^\rho + \alpha_2 \left( \frac{w_1}{\alpha_1} \right)^{\frac{1}{\rho}} \left( \frac{w_2}{\alpha_2} \right)^{\frac{1}{\rho}} x_1^\rho \right)^{\frac{1}{\rho}} \]

\[ = x_1 \left( \alpha_1 + \alpha_2 \left( \frac{w_1}{\alpha_1} \right)^{\frac{1}{\rho}} \left( \frac{w_2}{\alpha_2} \right)^{\frac{1}{\rho}} \right)^{\frac{1}{\rho}} \]

Solve for \( x_1 \) and arrange things symmetrically.
\[
x_1 = \frac{y_0}{\left( \alpha_1 + \alpha_2 \left( \frac{w_1}{\alpha_1} \right)^{\frac{\alpha}{1-\rho}} + \alpha_2 \left( \frac{w_2}{\alpha_2} \right)^{\frac{\alpha}{1-\rho}} \right)^{\frac{1}{\rho}}} = \frac{y_0 \left( \frac{w_2}{\alpha_2} \right) \frac{1}{\rho}}{\left( \alpha_1 \left( \frac{w_1}{\alpha_1} \right)^{\frac{\alpha}{1-\rho}} + \alpha_2 \left( \frac{w_2}{\alpha_2} \right)^{\frac{\alpha}{1-\rho}} \right)^{\frac{1}{\rho}}}
\]

This is the cost-minimizing, constant-output input demand function for the first input (labor), \(x_1^*(w_1, w_2, y_0)\). The corresponding demand function for the second input (capital) is

\[
x_2^*(w_1, w_2, y_0) = \frac{y_0 \left( \frac{w_1}{\alpha_1} \right) \frac{1}{\rho}}{\left( \alpha_1 \left( \frac{w_1}{\alpha_1} \right)^{\frac{\alpha}{1-\rho}} + \alpha_2 \left( \frac{w_2}{\alpha_2} \right)^{\frac{\alpha}{1-\rho}} \right)^{\frac{1}{\rho}}}.
\]

This is obtained either by using the symmetry of the problem or by substituting the solution for \(x_1^*\) into equation (1).

The firm’s cost function is \(c(w_1, w_2, y) = w_1 x_1^*(w_1, w_2, y) + w_2 x_2^*(w_1, w_2, y)\); that is, the value of the firm’s cost when the cost-minimizing values of the two inputs are used.

\[
c(w_1, w_2, y) = w_1 \left( \frac{y \left( \frac{w_1}{\alpha_1} \right)^{\frac{1}{\rho}}}{\left( \alpha_1 \left( \frac{w_1}{\alpha_1} \right)^{\frac{\alpha}{1-\rho}} + \alpha_2 \left( \frac{w_2}{\alpha_2} \right)^{\frac{\alpha}{1-\rho}} \right)^{\frac{1}{\rho}}} \right) + w_2 \left( \frac{y \left( \frac{w_2}{\alpha_2} \right)^{\frac{1}{\rho}}}{\left( \alpha_1 \left( \frac{w_1}{\alpha_1} \right)^{\frac{\alpha}{1-\rho}} + \alpha_2 \left( \frac{w_2}{\alpha_2} \right)^{\frac{\alpha}{1-\rho}} \right)^{\frac{1}{\rho}}} \right)
= y \left( \frac{w_1 \left( \frac{w_1}{\alpha_1} \right)^{\frac{\alpha}{1-\rho}} + w_2 \left( \frac{w_2}{\alpha_2} \right)^{\frac{\alpha}{1-\rho}}}{\left( \alpha_1 \left( \frac{w_1}{\alpha_1} \right)^{\frac{\alpha}{1-\rho}} + \alpha_2 \left( \frac{w_2}{\alpha_2} \right)^{\frac{\alpha}{1-\rho}} \right)^{\frac{1}{\rho}}} \right)
\]

The average total cost is obtained by dividing total cost by output and the marginal cost is the derivative of total cost with respect to output.

\[
ac(w_1, w_2, y) = \frac{c(w_1, w_2, y)}{y} = \frac{w_1 \left( \frac{w_1}{\alpha_1} \right)^{\frac{\alpha}{1-\rho}} + w_2 \left( \frac{w_2}{\alpha_2} \right)^{\frac{\alpha}{1-\rho}}}{\left( \alpha_1 \left( \frac{w_1}{\alpha_1} \right)^{\frac{\alpha}{1-\rho}} + \alpha_2 \left( \frac{w_2}{\alpha_2} \right)^{\frac{\alpha}{1-\rho}} \right)^{\frac{1}{\rho}}}
\]

\[
MC(w_1, w_2, y) = \frac{\partial c}{\partial y} = \frac{w_1 \left( \frac{w_1}{\alpha_1} \right)^{\frac{\alpha}{1-\rho}} + w_2 \left( \frac{w_2}{\alpha_2} \right)^{\frac{\alpha}{1-\rho}}}{\left( \alpha_1 \left( \frac{w_1}{\alpha_1} \right)^{\frac{\alpha}{1-\rho}} + \alpha_2 \left( \frac{w_2}{\alpha_2} \right)^{\frac{\alpha}{1-\rho}} \right)^{\frac{1}{\rho}}}
\]

Note that marginal cost and average cost are equal and do not depend on output. With a little algebra it is possible to verify the Lagrange multiplier, \(\lambda\), is identically equal to marginal cost. While this property holds for any production function, the constancy of marginal cost result holds only for homothetic production functions like the CES function.